L = test section length, ft.
T = wall temperature, °F.
V = mean velocity, ft./sec.
W = mass flow rate, lb./hr.
c_p = heat capacity at constant pressure, B.t.u./lb. °F.
g = gravitational acceleration, ft./sec.^{2}
h = heat transfer coefficient, B.t.u./hr. sq. ft. °F.
k = thermal conductivity, B.t.u./hr. ft. °F.
t = fluid temperature, °F.

Greek Letters

β = coefficient of volumetric expansion, °F.$^{-1}$
μ = dynamic viscosity, lb./ft. hr.
ν = kinematic viscosity, sq. ft./sec.
ρ = density, lb./cu. ft.
Δ = difference (that is Δt is temperature difference)

Subscripts

a = based on arithmetic average temperature difference
b = based on bulk fluid temperature
lm = based on logarithmic-mean-temperature difference
L = condition at distance L from entrance
o = condition at entrance
w = based on tube wall temperature

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Manuscript received January 25, 1960; revision received May 10, 1960; paper accepted May 10, 1960.

Turbulent Newtonian Flow in Annuli

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In recent years there have been quite a few experimental studies on turbulent flow in annuli. In this paper a Prandtl mixing-length approach is applied to give a friction factor vs. Reynolds number expression for annuli [see Equation (22) and Table 1]; this expression describes tube flow and slit flow as special cases. No new adjustable constants appear in the final result other than those determined earlier for tube flow. The final expression is found to predict friction factors within the accuracy of the existing experimental data. The mixing-length friction-factor expression is thus substantially more accurate than the usual hydraulic-radius procedure and of comparable accuracy to other recent annulus friction-factor treatments.

Simple empirical theories of turbulence have often been used to describe the relation between pressure drop and flow rate for the turbulent flow of Newtonian fluids in smooth tubes. In this paper the authors show how one of these theories, the Prandtl mixing-length model, can be applied quite successfully to the turbulent flow of Newtonian fluids in annuli.

DEFINITION OF FRICTION FACTOR AND REYNOLDS NUMBER

Pressure-drop data for the flow of Newtonian fluids in smooth concentric annuli can be correlated by the use of three dimensionless groups. In general these groups will be a friction factor, a Reynolds number, and a geometric factor a.

A general definition for the friction factor is (*1*)

$$F = A\,K f \tag{1}$$

For an annulus the total drag force on the entire wetted surface is $\Delta p \pi R^2 (1-a^2)$. The characteristic area may be taken as the total wetted area $2\pi RL(1+a)$. The characteristic kinetic energy per unit volume may be chosen to be $\frac{1}{2}\rho V^2$. Thus for annuli the friction factor is defined by

$$f = \frac{R(1-a)}{L}\,\frac{\Delta p}{\rho V^2} \tag{2}$$

This is the same friction factor which is obtained by use of the mean-hydraulic-radius and which is commonly used for annuli (*9, 12*).

For the isothermal, steady, axial flow of an incompressible fluid in a coaxial annulus integration of the equation of motion gives the following well-known expression for the flux of z-momentum in the r-direction (*1*, Equation 2.4-3):

$$\tau = \frac{R\Delta p}{2L}\left(\frac{r}{R} - \lambda^2 \frac{R}{r}\right) \tag{3}$$

This momentum-flux distribution is shown in Figure 1, where one can see that the momentum flux is negative at the inner wall ($r = aR$), that it becomes zero at some intermediate radial distance ($r = \lambda R$), and that it is positive at the outer wall ($r = R$). The quantity λ is defined as that value of $\xi = r/R$ for which the local velocity is a maximum and the momentum flux is zero.

For the laminar flow of Newtonian fluids $\tau = -\mu\,(du/dr)$; insertion of this expression into Equation (3) and integration gives the velocity distribution:

$$u = \frac{R^2\Delta p}{4\mu L}\left[1 - \xi^2 + \frac{1-a^2}{\ln(1/a)}\ln \xi\right] \tag{4}$$

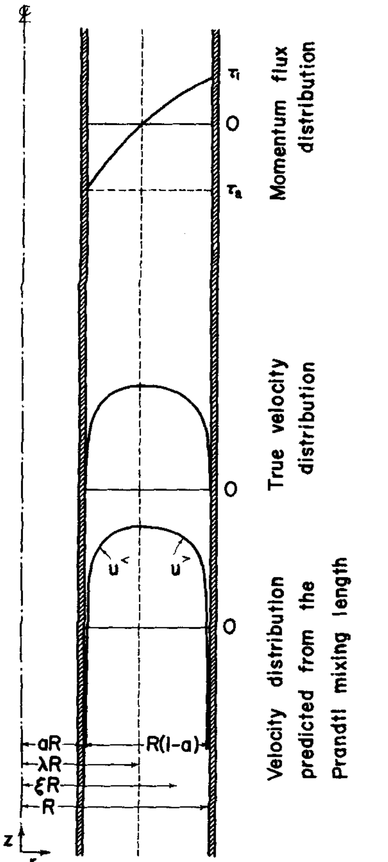

Fig. 1. Momentum flux distribution and velocity profile for axial annular flow.

It can be deduced from Equation (4) that u is a maximum in laminar flow at $r/R = \lambda$, where

$$\lambda = \sqrt{\frac{1 - a^2}{2 \ln (1/a)}} \qquad (5)$$

Values of λ are given in Table 1. The volume throughput and average velocity are found by integration of Equation (4) over the annulus:

$$V = \frac{q}{\pi R^2 (1 - a^2)} = \frac{R^2 \Delta p}{8 \mu L} \left[(1 + a^2) - \frac{(1 - a^2)}{\ln (1/a)} \right] \qquad (6)$$

Equations (2) and (6) may be combined to relate annulus friction factors and average flow velocities. One can force the expression for f to have the same form as that for circular tubes

$$f = 16/N_{Re}^{(a)} \qquad (7)$$

by defining the Reynolds number $N_{Re}^{(a)}$:

$$N_{Re}^{(a)} = \frac{D(1 - a) V \rho}{\mu} \phi \qquad (8a)$$

$$\phi = \frac{1}{(1 - a)^2} \left[(1 + a^2) - \frac{(1 - a^2)}{\ln (1/a)} \right] \qquad (8b)$$

where $D = 2R$. This Reynolds number is equal to the usual mean-hydraulic-radius Reynolds number multiplied by the factor ϕ, representative values of which are given in Table 1.

Use of this Reynolds number was presumably first proposed by Fredrickson and Bird (*6*). For the limiting case of tubes ($a = 0$) this Reynolds number reduces to $N_{Re}^{(0)} = DV\rho/\mu$ which is conventionally used in tube flow. For the limiting case of parallel plates ($a \rightarrow 1$) the Reynolds number defined in Equation (8) becomes $N_{Re}^{(1)} = 4R(1 - a)\, V\rho/3\mu$, where $R(1 - a)$ is the inter-plate distance. This Reynolds number, $N_{Re}^{(1)}$, is two-thirds times the customary mean-hydraulic-radius Reynolds number for parallel plate flow (*23*).

FRICTION FACTOR DATA

Experimental data taken on turbulent flow in annuli up to 1948 have been critically reviewed by Rothfus (*13*). Pressure-drop measurements on turbulent flow in annuli taken since that time have been reported by Rothfus, Monrad, and Senecal (*15*), Knudsen and Katz (*8*), and Walker, Whan, and Rothfus (*21*). The pressure drop data in tubes which have been taken by Nikuradse (*11*), supplemented by the transition region data of Senecal and Rothfus (*19*), are sufficient to establish f vs. $N_{Re}^{(0)}$ relations in the limiting case of $a = 0$. Pressure-drop measurements on flow between parallel plates, the limiting case of $a \rightarrow 1$, have been reported by Whan and Rothfus (*23*).

It was desired to bring together the friction-factor data reported above into one correlation, following the recommendation of Fredrickson and Bird (*6*). Data taken at various values of a from the sources listed in Table 2 were recalculated in terms of f and $N_{Re}^{(a)}$ and then plotted. After they were smoothed, cross plots were made of f vs. a at constant values of $N_{Re}^{(a)}$; a sample cross plot at $N_{Re}^{(a)}$ equal to 8,000 is shown in Figure 2. It can be seen that the friction factor at small values of a is about 10% greater than that at $a = 0$. The introduction of a small inner core thus affects the friction loss behavior of the empty tube quite markedly. As a is increased, the friction factor decreases steadily until its magnitude for the parallel plate case is somewhat below that for tube flow at the same Reynolds number.

APPLICATION OF PRANDTL THEORY

Empirical theories of turbulence have long been used to describe velocity-profile and pressure-drop behavior in turbulent flow in tubes. It was desired to see if such an approach could also represent pressure drops in turbulent flow in annuli. The Prandtl mixing-length model was chosen for investigation, as it is a simple model which leads to acceptably accurate velocity profiles, especially in the central regions of closed conduits at high Reynolds numbers (*3, 14, 15*).

Within experimental error it has been observed (*15, 16, 20*) that, in annular turbulent flow well beyond the laminar-turbulent transition, the maximum time-smoothed velocity occurs at the same radial distance as in laminar flow, that is λ given by Equation (5). The annulus may be divided into two regions, one region (denoted by the superscript $<$) between the inner wall and the radius of maximum velocity λR, the other region (denoted by the superscript $>$) between the radius of maximum velocity and the outer wall. From Equation (3) the momentum fluxes at the inner wall (τ_a) and the outer wall (τ_1) are given by

$$\tau_a = \frac{R\,\Delta p}{2L} \left(\frac{a^2 - \lambda^2}{a} \right);$$

$$\tau_1 = \frac{R\,\Delta p}{2L} (1 - \lambda^2) \qquad (9)$$

Friction velocities u_{*a} and u_{*1} may then be defined on the basis of the momentum fluxes at the two walls:

$$u_{*a} = \sqrt{-\tau_a/\rho}; \quad u_{*1} \sqrt{\tau_1/\rho} \qquad (10)$$

Dimensionless velocities $u^{+<}$ and $u^{+>}$ may be defined for each region as the ratio of the local velocity $u^<$ or $u^>$ to the appropriate friction velocity:

$$u^{+<} = u^</u_{*a}; \quad u^{+>} = u^>/u_{*1} \qquad (11)$$

Dimensionless distances from the inner and outer walls may also be defined respectively:

TABLE 1. VALUES OF λ, ϕ, G, AND H

a	λ, from Equation (5)	ϕ, from Equation (8)	G, from Equation (23)*	H, from Equation (24)*
0.00	0.0000	1.0000	4.000	0.400
0.05	0.4080	0.7419	3.747	0.293
0.10	0.4637	0.7161	3.736	0.239
0.15	0.5076	0.7021	3.738	0.208
0.2	0.5461	0.6930	3.746	0.186
0.3	0.6147	0.6820	3.771	0.154
0.4	0.6770	0.6759	3.801	0.131
0.5	0.7355	0.6719	3.833	0.111
0.6	0.7915	0.6695	3.866	0.093
0.7	0.8455	0.6681	3.900	0.076
0.8	0.8981	0.6672	3.933	0.060
0.9	0.9496	0.6668	3.967	0.046
1.0	1.0000	0.6667	4.000	0.031

* Based upon $\kappa^{>} = \kappa^{<} = 0.407$ and $C^{>} = 5.674$.

TABLE 2. SOURCES OF FRICTION-FACTOR DATA

Investigator	Ref.	Outer tube Material	Outer tube D (in.)	Inner tube Material	Inner tube aD (in.)	Geometric Ratio, a	Calming Length (multiples of $D(1-a)$)
Nikuradse	*11*		0.394 to 1.97			0	55 to 67
Senecal	*18, 19*	Brass	0.500, 0.750			0	96 or more
Walker	*20, 21*	Brass	0.750	Steel	0.0196	0.0260	148 or more
		Brass	0.750	Steel	0.0500	0.0667	154 or more
		Brass	0.750	Steel	0.0938	0.125	165 or more
		Brass	0.750	Steel	0.1240	0.165	173 or more
		Brass	0.750	Copper	0.2484	0.331	205 or more
		Brass	0.750	Copper	0.374	0.499	287 or more
Rothfus	*13, 15*	Brass	3.078	Brass	0.500	0.1625	82
		Brass	3.078	Brass	2.000	0.650	195
Knudsen	*7, 8*	Plexiglas	2.240	Copper	0.623	0.278	31
Kratz, et al.	*18*	Wrought iron	2.07	Wrought iron	1.30	0.629	71
Carpenter, et al.	*2*	Copper	0.834	Copper	0.625	0.750	230
Whan	*22, 23*	Brass rectangular passage, 14 in. wide by 0.700-in. clearance				1.000	154

$$y^{+<} = \frac{R(\xi - a)u_{*a}\rho}{\mu}; \qquad y^{+>} = \frac{R(1-\xi)u_{*1}\rho}{\mu} \quad (12)$$

The usual assumption is made that the laminar contribution to the momentum flux is negligible compared with the turbulent contribution. Then the total momentum flux may be given by the Prandtl mixing-length model

$$\tau = -\rho\, l^2 \left|\frac{du}{dr}\right| \frac{du}{dr} \quad (13)$$

neglecting curvature effects. The mixing-length is in turn generally assumed to be proportional to the distance from the wall; thus $l = \kappa^< R(\xi - a)$ near the inner wall and $l = \kappa^< R(1 - \xi)$ near the outer wall. Here, to be general, the authors have postulated two values of the proportionality constant: $\kappa^<$ for the inner region and $\kappa^>$ for the outer region. Equation (13) leads to the following relations between velocity gradient and momentum flux for the two regions of the annular space:

$$\tau = -\rho\, \kappa^{<2} R^2 (\xi - a)^2 (du^</dr)^2 \quad (14a)$$

for $a \leq \xi \leq \lambda$

$$\tau = +\rho\, \kappa^{>2} R^2 (1 - \xi)^2 (du^>/dr)^2 \quad (14b)$$

for $\lambda \leq \xi \leq 1$

Now the assumption [which Equation (3) shows is false] is made that the momentum flux everywhere in the inner region is equal to τ_a and everywhere in the outer region it is τ_1; this assumption is consistent with that made in Prandtl's original development for tubes. After introduction of the dimensionless velocity and distance parameters u^+ and y^+, Equations (14*a*) and (14*b*) may be integrated over their respective regions to give

$$u^{+<} = (1/\kappa^<) \ln y^{+<} + C^< \quad \text{for } a \leq \xi \leq \lambda \quad (15a)$$

$$u^{+>} = (1/\kappa^>) \ln y^{+>} + C^> \quad \text{for } \lambda \leq \xi \leq 1 \quad (15b)$$

These expressions are analogous to the Prandtl universal velocity distribution expression for tubes (9). Time-smoothed velocity distributions of this form are moderately valid in the interior portions of annuli, but they predict velocities approaching negative infinity (as does their analogue for tube flow) adjacent to the bounding surfaces (see Figure 1). Because of the anomalous behavior near the wall the skin friction cannot be calculated from the velocity gradient at the wall. The constants of integration $C^<$ and $C^>$ are related by the requirement of continuity of local velocity at $r = \lambda R$.

The volumetric rate of flow through the annulus is the sum of that through the two regions:

$$q = 2\pi R^2 \int_a^\lambda u^< \xi\, d\xi + 2\pi R^2 \int_\lambda^1 u^> \xi\, d\xi \quad (16)$$

Upon substitution for $u^<$ and $u^>$ from Equation (11) and Equations (15*a*) and (15*b*) and integration this becomes

$$q = \pi R^2 u_{*a}(\lambda^2 - a^2)\left[\frac{1}{\kappa^<}\ln\frac{R(\lambda - a)u_{*a}\rho}{\mu} - \frac{a}{\kappa^<(\lambda + a)} - \frac{1}{2\kappa^<} + C^<\right] + \pi R^2 u_{*1}(1-\lambda^2)\left[\frac{1}{\kappa^>}\ln\frac{R(1-\lambda)u_{*1}\rho}{\mu} - \frac{1}{\kappa^>(1+\lambda)} - \frac{1}{2\kappa^>} + C^>\right] \quad (17)$$

The constant $C^<$ may be expressed in terms of $C^>$ as explained above, and an over-all friction velocity u_{**} may be defined as $u_{**} = \sqrt{R\Delta p/2\rho L}$. Thus Equation (17) may be rewritten in a more convenient form:

$$q = \pi R^2 u_{**}\left\{(1-a^2)(1-\lambda^2)^{1/2}\left[\frac{1}{\kappa^>}\ln\frac{R(1-\lambda)(1-\lambda^2)^{1/2}u_{**}\rho}{\mu} + C^>\right] - B\right\} \quad (18)$$

where

$$B = \frac{(\lambda^2 - a^2)^{3/2}}{\kappa^< a^{1/2}}\left(\frac{a}{\lambda + a} + \frac{1}{2}\right) + \frac{(1-\lambda^2)^{3/2}}{\kappa^>}\left(\frac{1}{1+\lambda} + \frac{1}{2}\right) \quad (19)$$

Since for highly turbulent flow λ is dependent upon a [Equation (5)] and $\kappa^<$ and $\kappa^>$ are experimentally determined constants, the quantity B is a function of a only.

Substitution of the over-all friction velocity into the definition of f in Equation (2), followed by some rearrangement, gives

$$\sqrt{\frac{1}{f}} = \frac{V}{u_{**}} \cdot \frac{1}{\sqrt{2(1-a)}} = \frac{q}{\pi R^2 (1-a^2) u_{**} \sqrt{2(1-a)}} \quad (20)$$

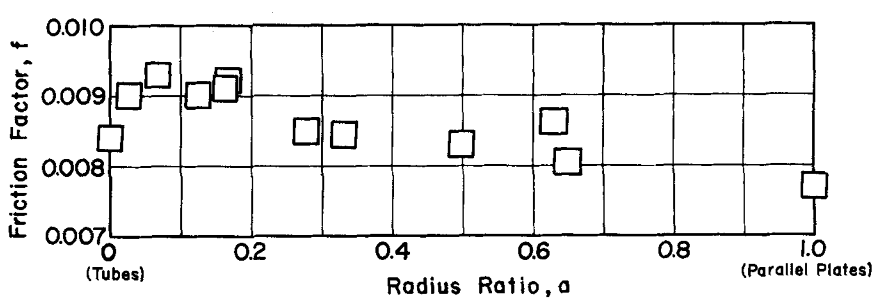

Fig. 2. Experimental friction factors vs. radius ratio for $N_{Re}{}^a = 8{,}000$.

Also it can be shown that the group $R(1-\lambda)(1-\lambda^2)^{1/2}\ u_{**}\rho/\mu$ of Equation (18) is related to $N_{Re}^{(a)}$ [Equation (8)] as follows:

$$\frac{R(1-\lambda)(1-\lambda^2)^{1/2}u_{**}\rho}{\mu} = N_{Re}^{(a)}\sqrt{f}$$

$$\frac{(1-\lambda)(1-\lambda^2)^{1/2}}{2\sqrt{2(1-a)}\left[\dfrac{1+a^2}{1-a}-\dfrac{1+a}{\ln(1/a)}\right]} \tag{21}$$

After one makes this substitution, a combination of Equations (18) and (20) leads to

$$\sqrt{\frac{1}{f}} = G\log_{10}(N_{Re}^{(a)}\sqrt{f}) - H \tag{22}$$

where G and H are given by

$$G = \frac{2.303\,(1-\lambda^2)^{1/2}}{\kappa^{>}\sqrt{2(1-a)}} \tag{23}$$

$$H = -\frac{(1-\lambda^2)^{1/2}}{\sqrt{2(1-a)}}\left(\frac{1}{\kappa^{>}}\ln\frac{(1-\lambda)(1-\lambda^2)^{1/2}}{2\sqrt{2(1-a)}\left[\dfrac{1+a^2}{1-a}-\dfrac{1+a}{\ln(1/a)}\right]}+C^{>}\right) + \frac{B}{(1-a^2)\sqrt{2(1-a)}} \tag{24}$$

Both G and H are functions of a alone.

EVALUATION OF PARAMETERS AND APPLICABILITY OF RESULTS

For the special case of tube flow ($a = 0$), B equals $3/2\kappa$ and H therefore is $-\dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\kappa}\ln\dfrac{1}{2\sqrt{2}}+C\right)+\dfrac{3}{2\kappa\sqrt{2}}$. This leads to a resistance law for tubes:

$$\sqrt{\frac{1}{f}} = \frac{2.303}{\kappa\sqrt{2}}\log_{10}(N_{Re}^{(0)}\sqrt{f}) + \frac{1}{\sqrt{2}}\left(\frac{1}{\kappa}\ln\frac{1}{2\sqrt{2}} - \frac{3}{2\kappa} + C\right) \tag{25}$$

A common representation of the experimental data for friction factors in smooth tubes is (*9, 11*)

$$\sqrt{\frac{1}{f}} = 4.0\log_{10}(N_{Re}^{(0)}\sqrt{f}) - 0.4 \tag{26}$$

Comparison of Equation (26) with Equation (25) gives $\kappa = 0.407$ and $C = 5.674$.

It is then arbitrarily assumed that these values of the mixing-length

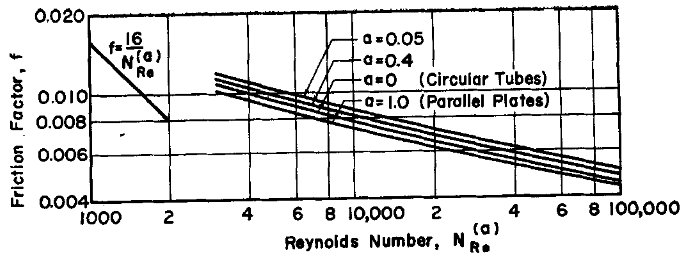

Fig. 3. Friction factor vs. Reynolds number for laminar and turbulent annular flow.

parameters κ and C are valid for the friction-factor expression for annuli as well as for tubes; that is $\kappa^{<} = \kappa^{>} = 0.407$ and $C^{>} = 5.674$. These values are not sufficiently accurate to predict annular velocity distributions; in annuli $\kappa^{<}$ is not equal to $\kappa^{>}$ but rather both $\kappa^{<}$ and $\kappa^{>}$ are functions of a (*9*, Figures 7.28 and 7.29). However $\kappa^{<}$ must approach $\kappa^{>}$ in the limiting case of parallel plates.

The quantities $G(a)$ and $H(a)$ based upon this arbitrary choice of $\kappa^{<}$, $\kappa^{>}$, and $C^{>}$ are given in Table 1. Plots of f vs. $N_{Re}^{(a)}$ for annuli derived from Equation (22) for representative values of a are shown in Figure 3. It can be seen that this theory predicts a jump in f as a small inner tube is introduced into an empty outer tube and also a shift in the curves downward as the parallel plate case is approached. Cross plots of f vs. a at constant Reynolds number, along with corresponding experimental data, are shown in Figure 4. The average absolute deviation of the experimental data points in Figure 4 from the predicted friction factors at the same a and $N_{Re}^{(a)}$ is 2.7%. Thus the curves calculated from this treatment predict friction factors with an accuracy comparable to the reproducibility of the experimental data.

For laminar flow in an annulus it can be shown analytically that the flow rate and friction factor converge to the empty tube value as the radius of the internal cylinder tends to zero (*5*). A similar convergence is expected for turbulent annular flow. As a result of the oversimplified treatment used here Equations (18) and (22) predict a spurious discontinuity in both q and f at $a = 0$. Nevertheless this treatment does lead to a satisfactory representation of friction-factor results over the range of a that has been studied experimentally.

PROCEDURE FOR USE OF CORRELATION

To calculate a value of pressure drop for a given average velocity in an annulus of radius ratio a the necessary steps are as follows: look up ϕ, G, and H in Table 1, calculate the Reynolds number $N_{Re}^{(a)}$ in Equation (8*a*), calculate f from Equation (22), and calculate Δp from Equation (2).

COMPARISON WITH PREVIOUS FRICTION-FACTOR CORRELATIONS

Other methods for obtaining annular friction factors are in common use. The oldest method consists of using a graph of f vs. N_{Re} for tubes to relate annular friction factors (Equation 2) with the mean-hydraulic-radius Reynolds number ($N_{Re}^{(a)}/\phi$) (*12*). Since the friction factors predicted by this method may be as much as 20% below the experimental values, especially in the range $0 < a < 0.5$ (see *21*, Figure 2), this method is not to be recommended.

A much more accurate treatment has been proposed by Rothfus and coworkers (*15, 21*), who applied the mean-hydraulic-radius concept to the fluid in the region $\lambda \leqq \xi \leqq 1$. Their empirical method and Equation (22) of this paper appear to represent the experimental data about equally well. The results of the present treatment are more conservative in that they predict very slightly higher friction factors than does the Rothfus method. The effort involved in using either of the two methods is nearly equal.

A third approach for obtaining annular friction factors is that of Deissler and Taylor (*4*). These authors, in a method which they have applied to eccentric as well as concentric annuli, locate λ by trial and error [at a position in general not that given by Equation (5)] on the assumption that the velocity distribution on both sides of λ is the same as that which has been experimentally determined for tubes at high Reynolds numbers. Although no thorough comparison of the results of their analysis with experimental fric-

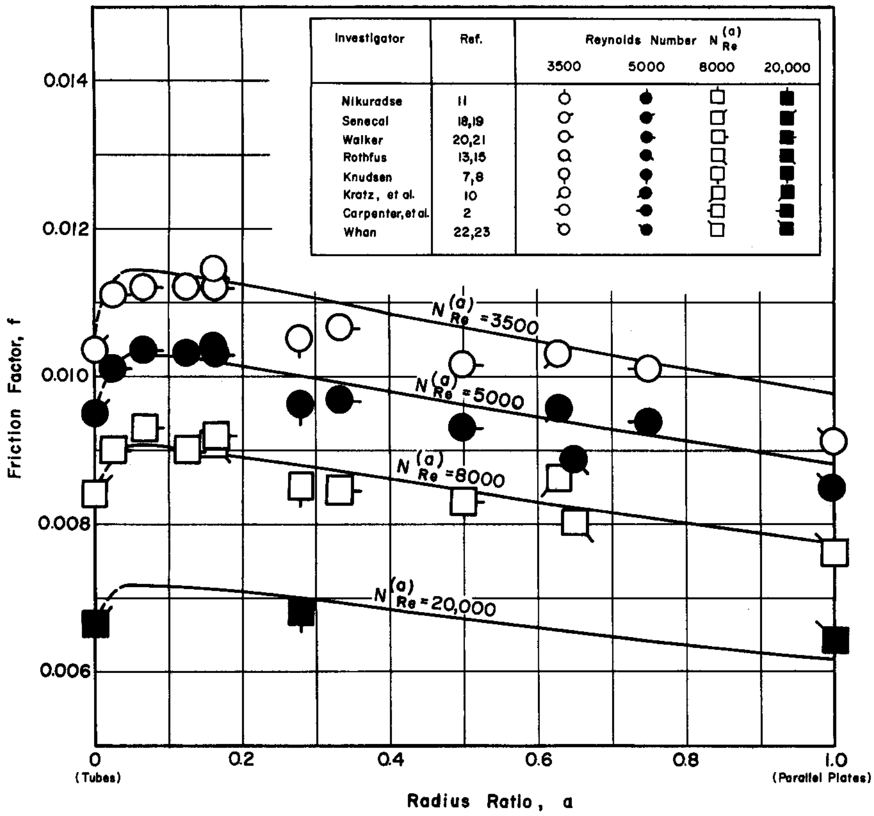

Fig. 4. Comparison of experimental and predicted friction factors for turbulent annular flow.

tion-loss data has been made, their results appear reasonable.

ACKNOWLEDGMENT

The authors desire to thank Mr. Richard N. Stae for collecting and analyzing the friction factor data which was used in the above analysis. In addition they are grateful for suggestions and comments made by Professors J. J. Barker, J. G. Knudsen, C. C. Monrad, and J. C. Slattery. Donald M. Meter wishes to gratefully acknowledge fellowship support provided by E. I. du Pont de Nemours and Company, Incorporated.

NOTATION

a = ratio of radius of inner tube to that of outer tube
A = characteristic area
B = function defined by Equation (19)
C = constant of integration, Equation (15)
D = diameter of outer tube of annulus
f = friction factor, Equation (2)
F = drag force on all wetted surfaces
G,H = functions defined by Equations (23) and (24)
K = characteristic kinetic energy per unit volume
l = Prandtl mixing length
L = length of tube or annulus
$N_{Re}^{(a)}$ = Reynolds number for annuli, Equation (8)
$N_{Re}^{(0)}$ = Reynolds number for tubes (special case of $N_{Re}^{(a)}$)
$N_{Re}^{(1)}$ = Reynolds number for parallel plates (special case of $N_{Re}^{(a)}$)
Δp = pressure drop across tube or annulus
q = volumetric rate of flow
r = radial coordinate in tube or annulus
R = radius of outer tube of annulus
u = local axial fluid velocity
u_{*a}, u_{*1} = friction velocity referred to inner or outer wall of annulus, Equation (9)
u_{**} = over-all friction velocity
u^+ = dimensionless velocity parameter
V = axial fluid velocity averaged over the cross section available for flow
y^+ = dimensionless distance from wall
z = axial coordinate in tube or annulus

Greek Letters

κ = mixing length proportionality constant
λ = dimensionless radial distance of maximum velocity in annulus
μ = fluid viscosity
ξ = dimensionless radial coordinate = r/R
π = 3.1416
ρ = fluid density
τ = rz-component of momentum flux, Equation (3)
τ_a, τ_1 = momentum flux at inner or outer wall of annulus, Equation (9)
ϕ = function of radius ratio, Equation (8b)

Superscripts

$<$ = region of annulus near inner wall
$>$ = region of annulus near outer wall

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Manuscript received January 18, 1960; revision received May 9, 1960; paper accepted May 10, 1960.